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Hand in on December 12th 2016 before the lecture

Advanced Statistical Physics II – Problem Sheet 8

Problem 1 – **Field-jump experiment**

(3P) Derive the response of an observable $A(t)$ for an external field that is switched on at time $t = 0$ and thus is of the form $h(t) = h_0\Theta(t)$. Assume an exponential correlation function $C(t) = C_0 e^{-t/\tau}$.

Problem 2 – **Fourier transformation of odd and even functions**

(3P) In the lecture the following identity for the real and imaginary part of a single-sided response function has been derived

$$\tilde{\chi}'(\omega) = \int \frac{dw'}{2\pi} i\tilde{\chi}''(\omega')\tilde{sig}(\omega - \omega') \tag{1}$$

Derive the analogous relation that gives the imaginary part $\tilde{\chi}''(\omega)$ in terms of the real part $\tilde{\chi}'(\omega)$.

Problem 3 – **Piezoelectric effect**

(3P) In the lecture we introduced the Hamiltonian for the piezoelectric effect, which reads as

$$H(q,p) = H_0(q,p) - E(t)P(q) - F(t)X(q)\,. \tag{2}$$

Here, $E(t)$ is a time-dependent electric field, $P(q)$ is the polarization, $F(t)$ is a time-dependent force and $X(q)$ is the extension. All these quantities are scalar quantities defined with respect to the x-coordinate.

Assume that there is a short electric pulse acting on the system at $t = 0$, hence $E(t) = E_0\delta(t)$ and the force, which is acting on the system with a constant value, is turned off at $t_1 > 0$, hence $F(t) = F_0(1 - \Theta(t - t_1))$.

Calculate the response of the quantities $P(t)$ and $X(t)$ in terms of the correlation functions $C_{PP}(t) = \langle P(t)P(0)\rangle$, $C_{PX}(t) = \langle P(t)X(0)\rangle$ and $C_{XX}(t) = \langle X(t)X(0)\rangle$.

Problem 4 – **Residue calculus for the inverse Fourier transform**

In the lecture the Fourier-transform of $f(t) = \Theta(t)e^{-t/\tau}$ has been shown to be $\tilde{f}(\omega) = \frac{1}{1/\tau - i\omega}$, when defining the Fourier-transform as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} f(t) \tag{3}$$

a) (2P) Show that this is actually a true statement.

b) (2P) In order to obtain the inverse Fourier transform of $\tilde{f}(\omega)$ the so called *residue theorem* is an efficient tool which will be described here briefly. Consider the contour integral

$$C = \oint dz \frac{1}{z} \tag{4}$$

along a closed path around the pole at $z = 0$. This can be solved when parameterizing a path as $z = Re^{it}$ for $t = [0, 2\pi]$ where R is the radius of the circle around $z = 0$. Then Eq. (**??**) is given by

$$C = \oint dz \frac{1}{z} = \int_0^{2\pi} \frac{1}{z(t)} \dot{z}(t) dt \tag{5}$$

$$= \int_0^{2\pi} \frac{1}{Re^{it}} iRe^{it} dt = 2\pi i \tag{6}$$

Show that

$$\oint dz \frac{1}{z^n} = \begin{cases} 2\pi i, n = 1 \\ 0, n \neq 1 \end{cases} \tag{7}$$

c) (2P) Another important identity is Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - z_0} = \text{Res}(\frac{f(z)}{z - z_0}, z_0), \tag{8}$$

which defines the residue of the function $\frac{f(z)}{z-z_0}$ at z_0.
Prove the left side of the equation by expressing $f(z)$ as a series around $f(z_0)$ and use the result from b)

Now you should be able to evaluate contour integrals in general.

d) (2P) As an example evaluate $\oint dz \frac{1}{1+z^2}$.

e) (3P) Obtain the inverse Fourier transform of $\tilde{f}(\omega) = \frac{1}{1/\tau - i\omega}$ defined as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-iwt} dw \tag{9}$$

with the help of the residue theorem.
Hint: Transform the inverse Fourier integral to a complex contour integral which goes from $-R$ to R and with an arc from R back to $-R$, where R goes to ∞. Why does the contribution from the arc go to zero when $R \to \infty$?