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Hand in on December 5th 2016 before the lecture

Advanced Statistical Physics II – Problem Sheet 7

Problem 1 – **Functional Taylor expansion**

The functional Taylor expansion for a functional $G[f]$ around a function $f_0(x)$ reads

$$G[f] = G[f_0] + \int \mathrm{d}x_1 \left.\frac{\delta G[f]}{\delta f(x_1)}\right|_{f=f_0} \big(f(x_1) - f_0(x_1)\big)$$
$$+ \frac{1}{2!}\int \mathrm{d}x_1 \mathrm{d}x_2 \left.\frac{\delta^2 G[f]}{\delta f(x_1)\delta f(x_2)}\right|_{f=f_0} \big(f(x_1) - f_0(x_1)\big)\big(f(x_2) - f_0(x_2)\big) + \dots \tag{1}$$

Let in the following be $f_0(x)$ an arbitrary function with $\lim_{x\to\pm\infty} f_0(x) = 0$.

a) (3P) Expand the functional $G_1[f] = \int \mathrm{d}x \left[f(x) + f(x)^2\right]$ around $f_0(x)$ to all non-vanishing orders.

b) (4P) Expand the functional $G_2[f] = \int \mathrm{d}x\, f'(x)^2$ around $f_0(x)$ to all non-vanishing orders.

Problem 2 –**Time-reversal symmetry**

The time correlation function of two observables $A(q,p)$ and $B(q,p)$ for a stationary system is defined as

$$C_{AB}(\tau) = \langle A(\tau)B(0)\rangle = \int \mathrm{d}p\mathrm{d}q\mathrm{d}q'\mathrm{d}q' A(q,p)B(q',p')\mathrm{e}^{-\tau L(q,p)}\delta(q-q')\delta(p-p')\rho_0(q',p') \tag{2}$$

Here, L is the Liouville operator and ρ_0 is a stationary density distribution.

a) (2.5P) Perform the substitution $p = -p^\star$ and $p' = -p'^\star$. This transformation corresponds to a time reversal of the process. Why? Assume that $\rho_0(q',p') = \rho_0(H(q',p'))$ is a function of the Hamiltonian only and that $H(q,p) \sim p^2$.

b) (1.5P) Give a relation between $C_{AB}(\tau)$ and $C_{AB}(-\tau)$ for the following observables:
i) $A = q$ and $B = p^2$
ii) $A = p$ and $B = H(q,p)$
iii) $A = p$ and $B = qp^3$

Problem 3 – **Inhomogenous differential equations**

We want to discuss two different methods to solve an inhomogeneous linear differential equation of first order.

a) (3P) Consider the inhomogeneous differential equation

$$\dot{x}(t) = p(t)\cdot x(t) + r(t). \tag{3}$$

First solve the homogeneous part and then derive a formula for the general solution of $x(t)$ with intial condition $x(t_0) = x_0$ by variation of the integration constant from the homogeneous solution.

b) (2P) As an application consider an electric circuit with a capacitor with capacity C and a resistor with resistance R connected to a battery of voltage V.

$$R\dot{Q}(t) + \frac{1}{C}Q(t) = V, \quad \text{where } Q(t=0) = 0. \tag{4}$$

Use the result from a) to solve for $Q(t)$.

c) (4P) Another helpful tool is to convert Eq. (4) to Fourier space, solve the equation in that space and back-transform to obtain the solution ofb the initial-value problem. The back-transform involves solving an integral of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\, dx = 2\pi i \sum_{Im(z_0)>0} \text{Res}\left(\frac{P(z)}{Q(z)} e^{iz}, z_0\right), \tag{5}$$

where $P(x)$ and $Q(x)$ are polynomials and z_0 is a pole of the integrand. The expression on the left-hand side can be evaluated when summing over every residue of Res $\left(\frac{P(z)}{Q(z)} e^{iz}, z_0\right)$ with z_0 in the upper complex half-plane. The residue for a complex function $f(z)$ can be calculated via the residue theorem

$$\oint_\gamma f(z)\, dz = 2\pi i \text{Res}(f, z_0), \tag{6}$$

where γ is a counterclockwise contour around the pole. The residue can either be evaluated by executing the contour-integral or using Cauchy's formula

$$\frac{n!}{2\pi i} \oint_\gamma \frac{g(z)}{(z - z_0)^{n+1}}\, dz = g^{(n)}(z_0)\,, \tag{7}$$

where $g^{(n)}$ is the n-th derivative of a function g.

Use the Fourier transformation to determine the solution of Eq. (4).