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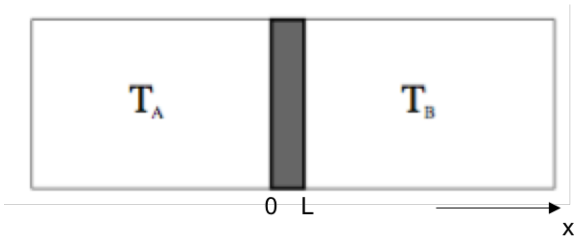
Hand in on November 21th 2016 before the lecture

Advanced Statistical Physics II – Problem Sheet 5

Problem 1 – **Heat diffusion**

Let two heat reservoirs R_A and R_B with temperatures T_A and T_B be connected by an heat-conducting solid, such that a heat current can flow from one reservoir to the other. Additionally, the reservoirs are isolated, so there is no other heat exchange with the surrounding, and so large that the heat current is not changing their temperatures. Furthermore, we assume the problem to be one-dimensional such that the solid has the length L and the interface between R_A and the solid is at $x = 0$. At $t = 0$ the temperature in the solid is $T(x, 0) = T_A$ and we have $T_A < T_B$ for $0 < x < L$.
Hint: We do not have mobile particles here. What does this mean for the substantial time derivative and the mass density $\rho(x, t)$?



a) (2P) The heat flux $J_q(x, t)$ through a solid is connected with the temperature gradient by the thermal conductivity coefficient k:

$$J_q(x,t) = -k\frac{\partial T(x,t)}{\partial x} . \tag{1}$$

Show that the temperature-field in the heat-conducting solid $T(x, t)$ is described by the equation

$$\frac{\partial T(x,t)}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 T(x,t)}{\partial x^2} , \tag{2}$$

Use the definition of the specific heat capacity c and the mass density ρ of the solid and the conservation equation for the energy from the lecture.

b) (1P) The entropy current J_s in the solid is given by $J_s = J_q/T$. Show with the help of the conservation equation for the entropy that the entropy production is given by

$$\sigma_s = k\left(\frac{1}{T}\frac{\partial T(x,t)}{\partial x}\right)^2 . \tag{3}$$

c) (2P) Find the stationary solution $T_s(x)$ of Eq. 2. Calculate the total entropy production in the solid in the stationary case, which is given as

$$P_s = A\int_0^L \mathrm{d}x\, \sigma(x,t) \tag{4}$$

d) (1P) We define the temperature profile $\tilde{T}(x, t)$ as $\tilde{T}(x, t) = T(x, t) - T_s(x)$. Rewrite Eq. 2 in terms of $\tilde{T}$. What are the initial condition $\tilde{T}(x, 0)$ and the boundary conditions $\tilde{T}(0, t)$ and $\tilde{T}(L, t)$ for the resulting equation?

e) (3P) Separate the time dependence from the space dependence by introducing $\tilde{T}(x,t) = A(t)B(x)$. Show that this leads to the equation

$$\frac{\mathrm{d}^2 B(x)}{\mathrm{d}x^2} = -\lambda B(x)\,. \tag{5}$$

Solve this equation and show that $\lambda = n^2\pi^2/L$ is the only non-trivial choice satisfying the boundary conditions.

f) (2P) Solve the resulting equation for $A(t)$ and write down the full result for $\tilde{T}(x,t)$ as an infinite sum the form

$$\tilde{T}(x,t) = \sum_{n=1}^{\infty} D_n A_n(t) B_n(x)\,. \tag{6}$$

Use the initial condition $\tilde{T}(x,0)$ to get an equation that determine the coefficients D_n.

g) (3P) Use the orthonormality of the trigonometric functions to compute the coefficients D_n and write down the full result for $T(x,t)$.

Problem 2 –**Functional derivatives and extremizing functionals**

A **functional** is a "function of functions", i.e. a mapping

$$P : M \longrightarrow \mathbb{R};\ f(x) \equiv f \longmapsto P[f]\,, \tag{7}$$

where M is some function space of interest. In the lecture we defined the functional derivative as

$$\frac{\delta P\left[f(\cdot)\right]}{\delta f(\tilde{x})} := \lim_{\epsilon\to 0} \frac{P\left[f(x) + \epsilon\delta(x-\tilde{x})\right] - P\left[f(x)\right]}{\epsilon}\,, \tag{8}$$

which is a function of $\tilde{x}$. Calculate the derivatives of the following functionals, with functions $f_i, g : \mathbb{R} \to \mathbb{R}$ such that the integral converges.

a) (1P)

$$P_1\left[f_1(\cdot)\right] = \int_{-\infty}^{\infty} \mathrm{d}x\, \left(f_1(x)\right)^n \tag{9}$$

b) (1P)

$$P_2\left[f_2(\cdot)\right] = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{a f_2(x)} \tag{10}$$

c) (1P)

$$P_3\left[f_3(\cdot)\right] = \exp\left(\int_{-\infty}^{\infty} \mathrm{d}x\, g(x) f_3(x)\right) \tag{11}$$

d) (1P)

$$P_4\left[f_4(\cdot)\right] = \int_{-\infty}^{\infty} \mathrm{d}x\, \left(f_4'(x)\right)^n \tag{12}$$

e) (2P) Let $L = L\left(f(x), f'(x), x\right)$ be a function of $f(x), f'(x)$ and x. Further let J be

$$J\left[f(\cdot)\right] = \int_{x_0}^{x_1} \mathrm{d}x\, L\left(f(x), f'(x), x\right) = 0 \tag{13}$$

Show that the function $f(x)$ that extremizes J with fixed boundaries x_0, x_1 fulfills the Euler-Lagrange equation

$$\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'} = 0\,. \tag{14}$$

Hint: Consider a multidimensional Taylor-series.