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Mathematical Tools in Statistical Mechanics

1) δ Function Calculus

For $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, calculate the following integrals

a)
$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x-b) f(ax), \tag{1}$$

b)
$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(ax-b) f(x), \tag{2}$$

b)
$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(g(x-b)) f(ax). \tag{3}$$

2 Gaussian Integrals

Prove that, for $a > 0$,

a)
$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{4}$$

b)
$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}. \tag{5}$$

3 Fourier Transform

For $q, b \in \mathbb{R}$ and $b > 0$, calculate the following Fourier transforms and extract the real and imaginary parts of your results.

a)
$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{ixq} e^{-x/b}, \tag{6}$$

b)
$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{ixq} e^{-x^2/b}. \tag{7}$$

4 Total Differentials

a) Calculate the total differential of the function
$$F(x, y, z,) = x^4 y^3 + zx + z^2 y. \tag{8}$$

b) Now, for which numerical value of $a \in \mathbb{R}$ is the following expression a total differential,
$$\mathrm{d}G = axyz\, \mathrm{d}x + x^2 z\, \mathrm{d}y + x^2\, \mathrm{d}z, \tag{9}$$
i.e. for which a does a function $G(x, y, z)$ exist?

5 Legendre Transform

Suppose we have a thermodynamic system for which we know the internal energy, $U(S,V) = S^2V^3$ with $\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V$. Legendre transform the internal energy to obtain the Helmholtz free energy $F(T,V) = U(S(T,V),V) - TS(T,V)$. Compute the total differential $\mathrm{d}F$ and compare the partial derivatives of $F(T,V)$ with the ones from $U(S,V)$.

6 Factorization

Factorize the following integral

$$I = \int \mathrm{d}x \int \mathrm{d}y \int \mathrm{d}z\, f(x)g(y)h(z)e^{h(x)+l(y)+m(z)}. \tag{10}$$

7 Logarithmic Calculus

For $f(a,b) = \ln(a^b)$, $g(a,b,c) = \ln(ab^c)$, $h(a,b,c) = \ln((a^b)^c)$ calculate the following partial derivatives

a)
$$\frac{\partial f}{\partial a}, \tag{11}$$

b)
$$\frac{\partial f}{\partial b}, \tag{12}$$

c)
$$\frac{\partial g}{\partial c}, \tag{13}$$

d)
$$\frac{\partial h}{\partial c}. \tag{14}$$

8 Lagrange Multipliers

A rope of length $L = 4\,\mathrm{m}$ is used to construct a rectangle with side lengths a and b, i.e. $L = 2a + 2b$. Using the method of Lagrange multipliers, calculate the values of a and b that maximizes the area $A = ab$. Can you also solve this problem without using the method of Lagrange multipliers?

9 Generating Function

Derive a general formula for the n-th moment

$$\langle x^n \rangle = \frac{\int dx x^n e^{-ax^2}}{\int dx e^{-ax^2}}, \tag{15}$$

using a generating field h.

10 Functional Derivative

When $f : \mathbb{R} \to \mathbb{R}$, consider the functional $I[f] = \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right)^m$ and compute functional derivative

$$\frac{\delta I[f(\cdot)]}{\delta f(\tilde{x})}. \tag{16}$$

11 Differential Equations

Solve the following ordinary linear differential equations,

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = af(x), \qquad \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} = af(x), \qquad \frac{\mathrm{d}f(x)}{\mathrm{d}x} = af(x) + g(x). \tag{17}$$