Freie Universität Berlin
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Statistical Physics and Thermodynamics (SS 2017)

Problem sheet 7

Hand in: Friday, June 9 during the lecture

http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre/

1 Equations of state (13 points)

Consider an ideal mono-atomic gas in 3 dimensions in the canonical (N, V, T) ensemble. The partition function for N indistinguishable gas atoms is

$$Z(N, V, T) = \frac{1}{N!}\left(\frac{V}{\lambda_t^3}\right)^N, \tag{1}$$

where $\lambda_t = h/\sqrt{2\pi m k_B T}$ is the thermal wavelength.

a) Calculate the free energy $F(N, V, T)$, the entropy $S(N, V, T)$ and the energy $U(N, V, T)$ as a function of the number of particles N, the volume V and the temperature T. **(4 points)**

b) Invert your results for the energy $U(N, V, T)$ to get an expression for the temperature as a function of N, V, U. **(1 point)**

c) Use your results from a) and b) to show that the entropy as a function of N, V, U is

$$S(N, V, U) = -k_B N \ln\left(\frac{N}{V}\right) + \frac{3}{2}k_B N \ln\left(\frac{U}{N}\right) + k_B N\left(\frac{3}{2}\ln\left(\frac{4\pi m}{3h^2}\right) + \frac{5}{2}\right). \tag{2}$$

How is this result related to the entropy in the microcanonical ensemble? **(2 points)**

Remark: This is known as the Sackur-Tetrode equation.

d) Use the Sackur-Tetrode equation and the differential first law of thermodynamics to derive $p(U, V, N)$ and the caloric equation of state $T(U, V, N)$. Explain why $p(U, V, N)$ is the same as the thermal equation of state $p(T, V, N)$. **(4 points)**

e) Use $U(N, V, T)$ and the thermal equation of state $p(T, V, N)$ to calculate $C_V = (\partial U/\partial T)_V$ and $C_p = (\partial U/\partial T)_p + p\,(\partial V/\partial T)_p$ for an ideal gas. **(2 points)**

2 Information entropy (7 points)

In information theory $I(p)$ is the information gained if an event, which has the probability p, occurs. If an event has a high probability one gains less information compared to an event with low probability. Furthermore the information from independently occurring events is linearly additive $I(p_1 p_2) = I(p_1) + I(p_2)$.

a) Find a function for the information $I(p)$, that is positive for all $p \in (0, 1)$ and linearly additive for independent events. **(1 point)**

b) Calculate the average over $I(p)$ for N independent events $i \in 1 \ldots N$, occurring with the probability p_i. How is this related to the rescaled entropy S/k_B we derived in the lecture? **(2 points)**

Now consider a generic sequence of N different characters, like the letters and special characters forming a text. From a given text one can estimate the probability p_i that a certain character i occurs.

c) How should one choose p_i so that S/k_B is maximal? Calculate S/k_B for a text containing N=27 different characters (a. . . z including spaces) all occurring with the same probability. **(3 points)**

Hint: For the determination of the maximum you can use a Lagrangian multiplier λ

$$\mathcal{L} = \frac{S}{k_B} + \lambda \left(\sum_i^N p_i - 1 \right) . \tag{3}$$

Maximize $\mathcal{L}$ *with respect to any probability* p_j *and calculate a value for* λ *by using the constraint* $\sum_i^N p_i = 1$.

Of course characters in an English text do not appear with an equal probability. By analyzing long English texts one can calculate the probability of each character[1]. Omitting dots, commas, etc. and using the natural logarithm, the rescaled entropy of English is $S_{\text{eng}}/k_B = 2.833$.

d) A measure of how efficiently information is stored in a text can be calculated by the so called redundancy R. The redundancy is defined as

$$R := 1 - \frac{S}{S_{max}}, \tag{4}$$

where S_{max} is the maximal entropy. R can take any value between zero (including zero) and one. If the redundancy is larger than 0 the *same* information can be expressed by reducing or combining different characters. Use your results from c) and the rescaled entropy of English to calculate the redundancy. **(1 point)**

Remark: Knowing the redundancy has some practical application such as data compression without loosing any information. On the other hand redundancy is also very beneficial. It allows one to understand what is said when only part of the information comes across, for example in a noisy room.

[1] Lee, E. Stewart. "Essays about Computer Security". University of Cambridge Computer Laboratory. p. 181.