Monday April 20, 2020
Due date: Sunday April 27, 2020

Prof. Dr. Roland R. Netz

Tutors: Cihan Ayaz (0.3.06)
*cihan.ayaz@fu-berlin.de*

Shane Carlson (0.3.34)
*shac87@zedat.fu-berlin.de*

Sina Zendehroud (0.3.33)
*sina.zendehroud@fu-berlin.de*

Please upload your solution as one pdf file on whiteboard.
Unreadable solutions and files submitted after the due date will get no points.

# Problem Set 1: Probability vs Intuition

## 1) The Gamma Function and the Stirling Formula

In this problem, we will derive the **Stirling formula** for factorials using the **Gamma function**, i.e. we will use a generalization of factorials to find an approximation for factorials.
The Gamma function is defined by the recurrence relations

$$\Gamma(1) = 1 \tag{1a}$$
$$\Gamma(x+1) = x\Gamma(x) \qquad \text{for } x \in \mathbb{R}_{>0} \tag{1b}$$

a) **(2P)** Use the recurrence relations in Equation (1) to prove that for $n \in \mathbb{Z}_{\geq 0}$ it follows

$$\Gamma(n+1) = n! \tag{2}$$

*Hint: Prove by induction.*

b) **(2P)** Show that the function

$$\Gamma(x) = \int_0^\infty \mathrm{d}y\, y^{x-1} e^{-y} \tag{3}$$

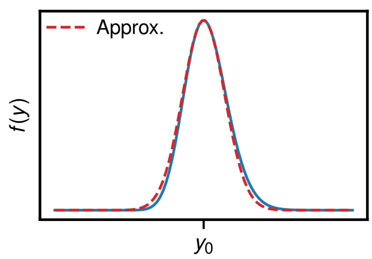
fulfills the recurrence relations in Equation (1), hence the Gamma function can be defined by the integral in Equation (3).

c) **(3P)** Prove the Stirling formula

$$n! \approx \sqrt{2\pi n}\, n^n\, e^{-n} \qquad \text{for } n \gg 1 \tag{4}$$

starting from $n! = \int_0^\infty \mathrm{d}y\, y^n e^{-y}$.

*Hint: If you plot the integrand $f(y) = y^n e^{-y}$, you will find that $f(y)$ has a sharp maximum at $y = y_0$ (see figure on the right). Rewrite the integrand as $f(y) = e^{g(y)}$, expand $g(y)$ to second order around $y = y_0$, and solve the integral. This is called a saddle-point approximation of the integral.*




d) **(1P)** Using your result from c), argue why

$$\ln n! \approx n \ln n - n \tag{5}$$

is a good approximation for $n \gg 1$.

## 2) The Birthday Problem

$N = 20$ students talk about their birthdays. For this problem neglect the existence of leap years, i.e., assume that a year has $M = 365$ days and that the probability to be born on a specific day is uniform.

a) **(1P)** What is the probability that all $N$ students have birthday on the same day?

b) **(2P)** What is the probability that all $N$ students have birthday on different days?

c) **(2P)** What is the probability that at least 2 students have birthday on the same day?

d) **(3P)** What is the number $N$ of students so that the probability that at least 2 students have birthday on the same day reaches 0.5?

## 3) Monty Hall Problem

You are a contestant on a game show where the prize is a new car. The game show host presents you with three closed boxes labeled A, B and C, and explains that two of them are empty, while one contains the keys to the new car. Your goal is to choose the box containing the keys and win the car.

a) **(1P)** After you have been presented the boxes and have made your choice, the host **randomly** (she/he doesn't know where the prize is!) opens one of the other boxes that you didn't chose and it **happens** to be empty. Let for example the pair $(A, B)$ denote the case where you choose $A$ and the host opens the box $B$ afterwards; the probability of this occurring is given by $P(A; B)$.

Assuming the prize is in box $A$, show that your probability of having chosen correctly is given by

$$P[\text{Win}] = \frac{P(A;B) + P(A;C)}{1 - P(B;A) - P(C;A)}. \tag{6}$$

What is the value of $P[\text{Win}]$?
*Hint: Remember that probabilities are normalized to one.*

b) **(1P)** Assume there is a probability $p$ that the host reveals the keys if you have chosen an empty box. Under this condition, you have chosen a box and the host opens a different box, revealing it to be empty. Show that the probability in equation 6, becomes

$$P[\text{Win}] = \frac{1}{3 - 2p}, \tag{7}$$

and discuss in brief the cases $p = 0$, $p = 0.5$ and $p = 1$.

c) **(1P)** In the classic Monty Hall problem, the format of the game is **always** as follows: you choose one of the three boxes, then the host opens one of the other two boxes, revealing it to be empty, and asks you if you would like to revise your choice or change to the other closed box.

Develop the best winning strategy using equation 7. Do you stick with the first choice or switch to the other closed box?

d) **(1P)** It is said that the famous mathematician Paul Erdős (1913 − 1996) was not convinced of the correct solution to problem c) until he was shown a simulation of this experiment. To convince your fellow students of your result, write a short simulation of one game where the contestant sticks with the initial choice and one where the contestant switches to the other closed box. Repeat the simulations and count how many times the contestant wins the car. Plot the number of wins versus the number of simulation runs for both cases in one plot. Without spoiling too much, your result should qualitatively look like the plot below.
To get the points for this part, put your code into the solution pdf file with a plot similar to the one below.

*Hint: Use a random number to place the prize in a box [1, 2, 3] (or select randomly from a list). Use the same procedure to pick a box as the contestant. Determine which boxes are empty and have not been picked by the contestant and choose randomly which one of them the game master opens. Then check if the initial choice yields the prize and increase the number of wins by one. For the switching case, obviously, you have to exchange the initial choice with the other box.*